A Lagrangian Approach to Optimal Lotteries in Non-Convex Economies

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Introduction

- Many economic models involve non-convex optimization problems.
  - Examples: private information models $\Rightarrow$ nonconvex incentive compatibility constraints.
  - Mathematically challenging to characterize, numerically difficult to solve.
- Lottery/Randomization/Mixed Strategy solutions to non-convex economies (Myerson, 1982; Prescott & Townsend, 1984; Arnott & Stiglitz, 1988):
  - Planner & agent choose probability distribution of action/consumption: convex problem in the probability space.
  - May increase the value of the objective function.
  - Real world correspondence: random audit, lottery in social programs, etc.
- Main difficulty for lottery problems: linear programming in high dimensional space.
- This paper: a new Lagrangian iteration algorithm to solve for optimal lotteries as weighted average of deterministic solutions.

This paper

- A new Lagrangian iteration algorithm to efficiently solve for optimal lotteries.
    1. Bridge pure strategy and lottery systems through Lagrangian iteration.
    2. Lottery solution is weighted average of unconstrained deterministic solutions along iteration.
- Theoretical guarantee: correctness and convergence (sub-gradient descent).
- Complexity estimate: orders of magnitude better than the linear programming approach.
- Applications: (1) Moral hazard, (2) Optimal tax with multi-dimensional hidden types.
- From applications: (1) much faster and memory-saving than conventional methods; (2) new insights when the randomized tax scheme is welfare improving.

Literature

Lagrangian Iteration Method

Illustrative Example: Moral Hazard Problem

- A continuum of representative agents take unobserved action $a \in A$, which affects output $q \in Q$ via $p(q|a)$. A and Q are finite sets. Agent consumption $c \in C$ (compact).

- Deterministic solution: the planner choose allocation $c(q)$ and recommend the agent to choose a, to solve

$$\max_{a, c(q)} \sum_q p(q|a) u(c(q), a), \tag{1}$$

subject to resource constraint & incentive compatibility constraint:

$$\begin{aligned} &\sum_q p(q|a)(c(q) - q) \leq 0; \\ &\sum_q p(q|a) u(c(q), a) \geq \sum_q p(q|\hat{a}) u(c(q), \hat{a}), \ \forall \hat{a} \in A \end{aligned} \tag{2}$$

- Problem can be highly non-convex.

Lottery Solution to Moral Hazard Problem

- Lottery problem: planner chooses $x \in \mathcal{P}(A \times C^{|Q|})$, $x = x(a, c(q_1), \cdots, c(q_{|Q|}))$ to:

$$\max \sum_{a \in A} \int_{c \in C^{|Q|}} x(a, dc) \sum_{q \in Q} p(q|a) u(c(q), a),$$

$$\text{s.t.} \sum_{a \in A} \int_{c \in C^{|Q|}} x(a, dc) \sum_{q \in Q} p(q|a)(c(q) - q) \leq 0$$

$$\int_{c \in C^{|Q|}} x(a, dc) \sum_{q} p(q|a) u(c(q), a) \geq \int_{c \in C^{|Q|}} x(a, dc) \sum_{q} p(q|\hat{a}) u(c(q), \hat{a}), \forall (a, \hat{a}) \in A \times A. \tag{3}$$

- This problem: linear in the probability space $\mathcal{P}(A \times C^{|Q|}) \Rightarrow$ directly solve with large-scale linear programming tools.

- **Challenge**: dimension of $\mathcal{P}(A \times C^{|Q|})$ is very high even for simple problem!

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Our Idea: construct lottery solution as weighted average of deterministic solutions.

New Method: Lagrangian Iteration for the Lottery Problem

Initial guess of: λ^0, γ^0. In the k-th iteration,

- Given λ^k, γ^k, update optimal allocation $(a^k, c^k) = \arg\max_{a,c} \mathcal{L}(a, c; \lambda^k, \gamma^k)$.
- Update Lagrangian multipliers λ^k and γ^k, with learning rate μ_k, e.g. $\mu_k = 1/k$:

$$\lambda^{k+1} = \max\left\{\lambda^k + \mu^k \sum_q p(q|a^k)(c^k(q) - q), 0\right\}.$$

$$\gamma^{k+1}_{\hat{a},a^k} = \max\left\{\gamma^k_{\hat{a},a^k} + \mu^k \left[\sum_q p(q|\hat{a})u(c^k(q), \hat{a}) - p(q|a^k)u(c^k(q), a^k)\right], 0\right\} \quad \forall \hat{a}$$

Intuition: Update Lagrangian multipliers according to how “close” the current allocation satisfies the inequality constraints $\Rightarrow$ sub-gradient descent.

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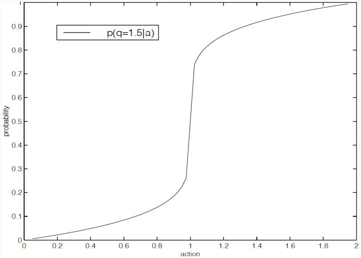
Final lottery solution:

$$x^N = \frac{1}{\sum_{k=1}^N \mu^k} \sum_{k=1}^N \mu^k \delta_{(a^k, c^k)}.$$

Calibration (Prescott, 1998)

- $u(a,c) = \sqrt{c} + 0.8\sqrt{2-a}$.
- Consumption set: $C = [0,2]$, output set $Q = \{0.5, 1.5\}$, action set: $A = 0.05 : \Delta a : 1.95$, with $\Delta a = 0.025$. Output distribution as function of action:

$$p(q = 1.5 \mid a) = \begin{cases} \frac{1-(1-a)^{0.2}}{2}, & \text{if } a < 1, \\ \frac{1+(a-1)^{0.2}}{2}, & \text{if } a \geq 1. \end{cases}$$



- Solution (Prescott, 1998):

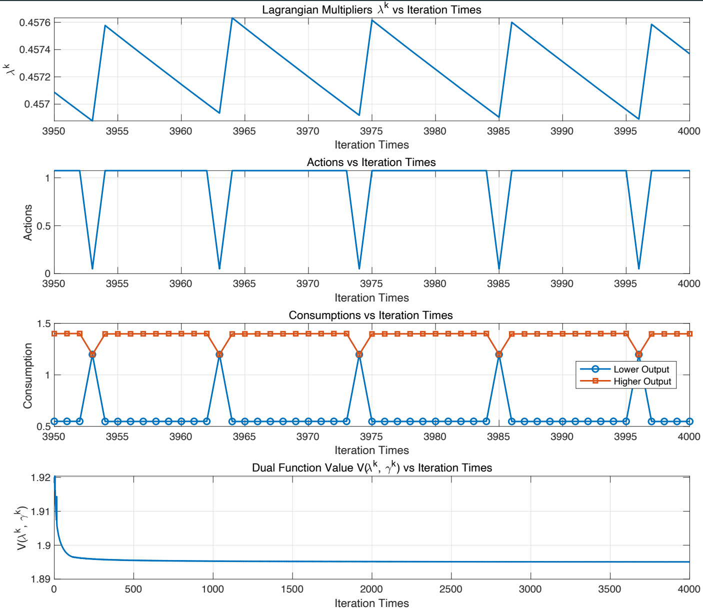
$$\pi(a = 0.05) = 0.0924,\ \pi(a = 1.075) = 0.9076$$

$$\pi(c = 1.20 \mid q = 0.5, a = 0.05) = 1,\ \pi(c = 1.19 \mid q = 1.5, a = 0.05) = 1.$$

$$\pi(c = 0.54 \mid q = 0.5, a = 1.075) = 0.5311,\ \pi(c = 0.55 \mid q = 0.5, a = 1.075) = 0.4689.$$

$$\pi(c = 1.40 \mid q = 1.5, a = 1.075) = 1.$$

Lagrangian multiplier λ^k, allocation a^k, c^k, dual value $V(\lambda^k, \gamma^k)$ along iteration



Theoretical Framework

General Framework

Deterministic problem with a continuum of agents, action $a \in A$ (finite), consumption $c \in C$ (compact), payoff function f:

$$\begin{aligned} &\max_{a \in A,\, c \in C} f(a,c), \\ \text{s.t. }\ &\text{resource constraints } g_i(a,c) \le 0 \quad i \in \{1,\dots,m\}, \\ &\text{incentive constraints } h_j(a,c) \le 0 \quad j \in \{1,\dots,\ell\}, \end{aligned} \tag{4}$$

Lottery problem with probability $x(a,dc) \in \mathcal{P}(A \times C)$:

$$\begin{aligned} &\max_{x \in \mathcal{P}(A\times C)} \sum_{a\in A} \int_{c\in C} f(a,c)x(a,dc), \\ \text{s.t. }\ &\sum_{a\in A} \int_{c\in C} g_i(a,c)x(a,dc) \le 0 \quad \forall i \in \{1,\dots,m\}, \\ &\int_{c\in C} h_j(a,c)x(a,dc) \le 0 \quad \forall a \in A,\ j \in \{1,\dots,\ell\}. \end{aligned} \tag{5}$$

Lagrangian Iteration: General Setup

Given $\lambda_i^1 (i \in \{1, \cdots, m\})$, $\gamma_{j,a}^1 (a \in A,\, j \in \{1, \cdots, \ell\})$, $\mu^k \in \mathbb{R}_+$, $N \in \mathbb{N}_+$. For $k = 1 : N$,

Step 1. Solve the Lagrangian problem.

$$(a^k, c^k) \in \arg \max_{a \in A, c \in C} \mathcal{L}(a, c; \lambda^k, \gamma^k).$$

Step 2. Update the Lagrangian multipliers.

$$\lambda_i^{k+1} = \max\{\lambda_i^k + \mu^k g_i(a^k, c^k), 0\}, \, \forall i \in \{1, \cdots, n\}.$$

$$\gamma_{j,a^k}^{k+1} = \max\{\gamma_{j,a^k}^k + \mu^k h_j(a^k, c^k), 0\}, \, \forall j \in \{1, \cdots, \ell\}.$$

$$\gamma_{j,a}^{k+1} = \gamma_{j,a}^k, \, \forall j \in \{1, \cdots, \ell\}, a \neq a^k.$$

Step 3. Construct the lottery solution with $\delta_{(a^k, c^k)}$ as $\delta-$measure at the point (a^k, c^k).

$$x^N := \frac{1}{\sum_{k=1}^N \mu^k} \sum_{k=1}^N \mu^k \delta_{(a^k, c^k)},$$

Main Theorem

Theorem
Suppose the sequence of learning rates $(\mu^k)_{k=1}^{\infty}$ satisfies

$$\sum_{k=1}^{\infty} \mu^k = \infty \text{ and } \sum_{k=1}^{\infty} (\mu^k)^2 < \infty.$$

Let x^* be the solution to the lottery problem, and suppose the corresponding Lagrangian multipliers to x^* exist. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}_+$, such that when $n > N$, x^n obtained from the Algorithm is an $\epsilon-$optimal solution to the problem.

Note: $\epsilon-$optimal solution is the solution that maximizes the same objective function subject to ϵ relaxation of the objective & constraints such as $\sum_{a \in A} \int_{c \in C} g_i(a,c)x(a,dc) \leq \epsilon$.

Thm 1a: given λ, γ, deterministic/lottery Lagrangians have same optimal value

Lagrangian in the pure strategy space $A \times C$:

$$\mathcal{L}(a, c; \lambda, \gamma) := f(a, c) - \sum_{i=1}^{n} \lambda_i g_i(a, c) - \sum_{j=1}^{\ell} \gamma_{j,a} h_j(a, c). \tag{6}$$

Lagrangian in the probability space $\mathcal{P}(A \times C)$:

$L(x; \lambda, \gamma) :=$

$$\sum_{a \in A} \int_{c \in C} f(a, c) x(a, dc) - \sum_{i=1}^{n} \lambda_i \sum_{a \in A} \int_{c \in C} g_i(a, c) x(a, dc) - \sum_{j=1}^{\ell} \sum_{a \in A} \gamma_{j,a} \int_{c \in C} h_j(a, c) x(a, dc), \tag{7}$$

Given Lagrangian multipliers λ, γ, we prove:

$$\max_{x \in \mathcal{P}(A \times C)} L(x; \lambda, \gamma) = \max_{a \in A, c \in C} \mathcal{L}(a, c; \lambda, \gamma). \tag{8}$$

Thm 1b: given λ, γ, optimal lottery only contains optimal deterministic solutions

Furthermore, if we define $Z = \arg\max_{a \in A, c \in C} \mathcal{L}(a, c; \lambda, \gamma)$, then

$$x^* \in \arg \max_{x \in \mathcal{P}(A \times C)} L(x; \lambda, \gamma)$$

if and only if

$$(a, c) \in Z \text{ a.s. with respective to the probability measure } x^*. \tag{9}$$

Thm 2: equivalence of lottery solution to the dual of deterministic problem

Theorem

The optimal objective value of the lottery problem is the same as the optimal objective value of the dual problem of the deterministic problem:

$$\max_{x\in\mathcal{P}(A\times C)} \min_{(\lambda,\gamma)\in\mathbb{R}_+^m\times\mathbb{R}_+^{\ell|A|}} L(x;\lambda,\gamma) = \min_{(\lambda,\gamma)\in\mathbb{R}_+^m\times\mathbb{R}_+^{\ell|A|}} \max_{a\in A,\, c\in C} \mathcal{L}(a,c;\lambda,\gamma).$$

$$\geq \max_{a\in A,\, c\in C} \min_{(\lambda,\gamma)\in\mathbb{R}_+^m\times\mathbb{R}_+^{\ell|A|}} \mathcal{L}(a,c;\lambda,\gamma).$$

Note: LHS is the primal form of the lottery problem:

$$\max_{x\in\mathcal{P}(A\times C)} \min_{(\lambda,\gamma)\in\mathbb{R}_+^m\times\mathbb{R}_+^{\ell|A|}} L(x;\lambda,\gamma) = \sum_{a\in A}\int_{c\in C} f(a,c)x^*(a,dc)$$

where x^* is the solution to lottery problem.

Remark

- The theorems above bridge the gap between the dual deterministic problem and the lottery problem.
- Theorem 2 motivates the Lagrangian iteration algorithm: solves the dual deterministic problem via sub-gradient descent.
- Now we'll prove that the algorithm indeed converges to the ($\epsilon-$optimal) lottery solution.

  1. If the Lagrangian iteration method converges, the solution we construct must
     1. Satisfy all the constraints
     2. Achieve the optimal value
  2. Following the sub-gradient descent literature, the algorithm we design must converge.

Key Proof of Main Theorem: given convergence, solution satisfies constraints

Want to show

$$\sum_{a\in A}\int_{c\in C} g_i(a,c)x^{\epsilon}(a,dc) \leq \epsilon, \quad \text{for } i \in \{1,\cdots,m\}.$$

For $i \in \{1,...,m\}$, we have

$$\sum_{a\in A}\int_{c\in C} g_i(a,c)x^{n}(a,dc) = \frac{1}{\sum_{k=1}^{n}\mu_k}\sum_{k=1}^{n}\mu_k g_i(a^k,c^k).$$

By the updating rule for λ_i, written

$$\lambda_i^{k+1} = \max\{\lambda_i^k + \mu^k g_i(a^k,c^k), 0\} \geq \lambda_i^k + \mu^k g_i(a^k,c^k), \quad k = 1,\cdots,n,$$

Proof: given convergence, solution satisfies the inequality constraints

$$\sum_{k=1}^{n} \lambda_i^{k+1} \geq \sum_{k=1}^{n} \left[\lambda_i^k + \mu^k g_i(a^k, c^k)\right],$$

which can be simplified as

$$\lambda_i^{n+1} \geq \lambda_i^1 + \sum_{k=1}^{n} \mu_k g_i(a^k, c^k) = \lambda_i^1 + \left(\sum_{k=1}^{n} \mu^k\right) \sum_{a \in A} \int_{c \in C} g_i(a, c) x^n(a, dc).$$

Hence

$$\sum_{a \in A} \int_{c \in C} g_i(a, c) x^n(a, dc) \leq \frac{\lambda_i^{n+1} - \lambda_i^1}{\sum_{k=1}^{n} \mu^k}. \tag{10}$$

Computational Complexity Analysis

- Take $\mu^k \sim k^{\frac{1}{2}(1+\rho)}$ for $0 < \rho < 1$. Then for $\epsilon > 0$, the overall computational complexity for finding an ϵ-optimal lottery solution with Lagrangian iteration is

$$O\left(\left(\frac{\frac{M}{\rho} + \bar{\Lambda}}{\epsilon}\right)^{\frac{2}{1-\rho}} |A||\hat{C}|(m+\ell)^{1+\frac{1}{1-\rho}}\right)$$

  where $\hat{C}$ is the discretized set of C.

- Comparing the complexity to that of the linear programming interior point method, we see in the case that $|\hat{C}| \sim |A| \sim \ell \gg m$,

$$|A||\hat{C}|(m+\ell)^{1+\frac{1}{1-\rho}} \sim |A|^{3+\frac{1}{1-\rho}} \ll |A|^7 \sim (|A||\hat{C}| + \ell|A| + m)^{3.5},$$

  for ρ close to 0.

- Bigger computational advantage if optimal deterministic c can be solved by FOC.

Computational Performance for Moral Hazard Problem with Different Δa

Δa	Iterations	CPU time (s)	LP CPU time (s)	Size of LP		
				#variable	#EC	#IC
0.2	500	0.006	0.05	4020	21	91
0.1	1000	0.01	0.13	8040	41	381
0.05	2000	0.02	0.44	15678	79	1483
0.025	4000	0.05	1.33	30954	155	5853
0.0125	8000	0.16	6.84	61506	307	23257
0.00625	16000	0.98	-	122610	611	92721

Note: $\Delta a = 0.0125$, it takes LP method 6.84 s to solve (2-3 orders of magnitude slower). LP cannot handle $\Delta a < 0.0125$ on a laptop due to memory limit. #EC & #IC: number of equality constraints and inequality constraints.

Application II: Optimal Tax with Multi-dimensional Heterogeneity

Optimal Taxation with Multi-dimensional Heterogeneity

- Agent preference with two-dimensional hidden types in productivity and labor supply elasticity w_h, η_h:

$$u_h(c, y) = \log(c) - \psi \frac{\left(\frac{y}{w_h}\right)^{\frac{1}{\eta_h}+1}}{\frac{1}{\eta_h} + 1}.$$

- Following Judd et al (2017), we choose five values of $w_h \in \{1, 2, 3, 4, 5\}$ and five values of $\eta_h \in \{\frac{1}{8}, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}, 1\}$, yielding 25 distinct types indexed by (w_h, η_h).
- We have $25 \times 24 = 600$ incentive constraints.
- Want to solve optimal lottery over (c, y), which is equivalent to optimal taxation with income tax schedule $T(y)$ and let each agent choose labor supply $\ell = \frac{y}{w_h}$ and consumption $c = y - T(y)$.

Optimal Income Tax with Hidden Types: Welfare Gain from Lotteries

(w, η)	Deterministic solution (c, y)	Lottery solution c	lottery solution y
1, 1	1.68, 0.42	1.697	0.43
1, $\frac{1}{2}$	1.77, 0.62	1.789	0.63
...	...	...	
3, 1	2.2, 1.83	2.456	(1.74 78.08%) (3.6, 21.92%)
3, $\frac{1}{2}$	2.47, 2.49	2.611	(2.62, 95.21%) (3.6, 4.79%)
...	...	...	
...	...	...	

Table 1: Optimal deterministic allocation versus optimal lottery solution.

Lottery scheme reduces 3.46% welfare loss of deterministic solution due to information friction (defined as Hicksian "compensating variation in resources" from full-information problem to achieve the same level of welfare).

Conclusion

- A new Lagrangian iteration algorithm to efficiently solve for optimal lotteries.
- Theoretical guarantee: correctness and convergence (sub-gradient descent).
- Complexity estimate: orders of magnitude better than the conventional approach.
- Applications: (1) Moral hazard problem, (2) Optimal income tax with hidden types.
- From applications: (1) much faster and memory-saving than conventional methods; (2) new insights when the randomized tax scheme is welfare improving.

Appendix

Related Literature

- Lottery solution to problems with non-convex constraints.
  1. Myerson (1982), Prescott and Townsend (1984a, 1984b), Arnott and Stiglitz (1988)
  2. Prescott (2004), Prescott and Townsend (2006), Doepke and Townsend (2006)
- Computational methods for moral hazard and optimal tax with hidden types:
  1. Su and Judd (2007), Armstrong et al. (2010), etc.
  2. Weiss (1976), Brito et al. (1995), Hellwig (2007), Gauthier and Laroque (2014), Judd et al (2017), among many others.
- Math literature on sub-gradient descent: Shor (2012); Nedic and Bertsekas (2001)

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